

# Z-Transform PML Algorithm for Truncating Metamaterial FDTD Domains

Omar Ramadan

Computer Engineering Department, Eastern Mediterranean University,  
Gazi Magusa, Mersin 10,  
Turkey  
Tel: + 90 392 630 1194 Fax: + 90 392 365 0711  
e-mail:omar.ramadan@emu.edu.tr.

**Abstract**—Efficient and simple formulations of the Perfectly Matched Layer (PML) are presented for truncating metamaterial Finite Difference Time Domain (FDTD) grids. The formulations are based on incorporating the Z-transform theory into the FDTD algorithm to model the frequency dependence property of the metamaterials. Numerical example carried out in one dimensional metamaterial domain is included to validate the proposed formulations.

**Index Terms**—Metamaterial, finite difference time domain, perfectly matched layer, Z-transform.

## I. INTRODUCTION

In recent years, the electromagnetic metamaterials with simultaneously negative electric permittivity and magnetic permeability [1] have been received much attention due to their unusual electromagnetic properties. These materials, referred as Negative Index Materials (NIM), have been realized by using different models such as the cold plasma, Drude medium and Lorentzian medium models. Recently, the Finite Difference Time Domain (FDTD) method [2] has been used successfully in simulating electromagnetic wave propagation in domains that have metamaterial properties [3]-[8].

When the FDTD method is used for modelling open region metamaterial domains, efficient Absorbing Boundary Conditions (ABCs) are needed to truncate the computational domain. Berenger's Perfectly Matched Layer (PML) [9] has been shown to be one of the most effective FDTD ABCs. Very recently, different PML formulations have been successfully introduced for truncating metamaterial FDTD computational domains [10]-[12]. In these formulations, the auxiliary differential equation (ADE) method and the bilinear frequency approximation technique have been incorporated into the FDTD implementations of the metamaterial.

In this paper, alternative PML formulations are presented for truncating metamaterial domains. The formulations incorporate the Z-transform theory [13] into the FDTD implementation of the frequency dependence of the metamaterials. The method has the advantage of simplicity as it allows direct FDTD implementations Maxwell's equations in the metamaterial domains. Numerical example carried out in one-dimensional (1-D) domain composed entirely of Lorentzian type metamaterial is included to validate the proposed formulations.

The paper is organized as follows. In section 2, the formulations of the proposed method are presented. In section 3, numerical results are included to validate the proposed formulations. Finally, conclusions are included in section 4.

## II. FORMULATION

Using the PML formulations of [12], the normalized field equations for a  $z$ -polarized transverse electromagnetic (TEM) wave propagating through metamaterial domain along the  $x$ -direction can be written in the frequency domain as

$$j\omega\varepsilon_r(\omega)S_x\tilde{E}_z = c\frac{\partial}{\partial x}\tilde{H}_y \quad (1)$$

$$j\omega\mu_r(\omega)S_x\tilde{H}_y = c\frac{\partial}{\partial x}\tilde{E}_z \quad (2)$$

where  $\tilde{E}_z$ , and  $\tilde{H}_y$  are the Fourier transform of  $E_z$  and  $H_y$  fields, respectively,  $\varepsilon_r(\omega)$  and  $\mu_r(\omega)$  are, respectively, the permittivity and permeability of the metamaterial, and  $S_x$  is the PML parameter defined as [12]

$$S_x = 1 + \frac{\sigma_x}{j\omega\varepsilon_0\varepsilon_r(\omega)} \quad (3)$$

where  $\sigma_x$  is the PML conductivity profile along the  $x$ -direction. In this paper,  $\varepsilon_r(\omega)$  and  $\mu_r(\omega)$  are assumed to be identical and realized with a Lorentz medium model [4] given by the following expression:

$$\varepsilon_r(\omega) = \mu_r(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\Gamma} \quad (4)$$

where  $\omega_p$  is the plasma frequency,  $\omega_0$  is the resonance frequency, and  $\Gamma$  is the absorption parameter of the medium. Substituting (3) and (4) into (1) and (2), the following can be obtained

$$j\omega\tilde{E}_z + j\omega\tilde{D}_z + \frac{\sigma_x}{\varepsilon_0}\tilde{E}_z = c\frac{\partial}{\partial x}\tilde{H}_y \quad (5)$$

$$j\omega\tilde{H}_y + j\omega\tilde{B}_y + \frac{\sigma_x}{\varepsilon_0}\tilde{H}_y = c\frac{\partial}{\partial x}\tilde{E}_z \quad (6)$$

where  $\tilde{D}_z$  and  $\tilde{B}_y$  are given by

$$\tilde{D}_z = \kappa(\omega)\tilde{E}_z \quad (7)$$

$$\tilde{B}_y = \kappa(\omega)\tilde{H}_y \quad (8)$$

where  $\kappa(\omega)$  is given by

$$\kappa(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\Gamma} \quad (9)$$

By using the Fourier transform relation  $j\omega \rightarrow \partial/\partial t$ , (5) and (6) can be written in the time domain as

$$\frac{\partial}{\partial t} E_z + \frac{\partial}{\partial t} D_z + \frac{\sigma_x}{\varepsilon_o} E_z = c \frac{\partial}{\partial x} H_y \quad (10)$$

$$\frac{\partial}{\partial t} H_y + \frac{\partial}{\partial t} B_y + \frac{\sigma_x}{\varepsilon_o} H_y = c \frac{\partial}{\partial x} E_z \quad (11)$$

Equations (10) and (11) can be discretized by using the FDTD algorithm [2] as

$$\begin{aligned} E_{z_i}^{n+1} &= r_{0_i} E_{z_i}^n - r_{1_i} (D_{z_i}^{n+1} - D_{z_i}^n) \\ &+ r_{1_i} \chi (H_{y_{i+1/2}}^{n+1} - H_{y_{i-1/2}}^{n+1}) \end{aligned} \quad (12)$$

$$\begin{aligned} H_{y_{i+1/2}}^{n+1} &= r_{0_{i+1/2}} H_{y_{i+1/2}}^n - r_{1_{i+1/2}} (B_{y_{i+1/2}}^{n+1} - B_{y_{i+1/2}}^n) \\ &+ r_{1_{i+1/2}} \chi (E_{z_{i+1}}^{n+1} - E_{z_i}^{n+1}) \end{aligned} \quad (13)$$

where  $\Delta_t$  is the time step,  $\Delta$  is the space cell size,  $\chi = c\Delta_t/\Delta$  and

$$r_{0_i} = \frac{1 - p_i}{1 + p_i}, \text{ and } r_{1_i} = \frac{1}{1 + p_i} \quad (14)$$

with

$$p_i = \frac{\sigma_{x_i} \Delta_t}{2\varepsilon_o} \quad (15)$$

In (12) and (13),  $D_{z_i}^{n+1}$  and  $B_{y_{i+1/2}}^{n+1}$  can be computed easily from (7) and (8), respectively, by using the Z-transform theory [13]. Hence, (9) can be written in the Z-domain [13] as

$$\kappa(\mathcal{Z}) = \frac{b_0 \mathcal{Z}^{-1}}{1 + a_0 \mathcal{Z}^{-1} + a_1 \mathcal{Z}^{-2}} \quad (16)$$

where  $\mathcal{Z}$  is the Z-transform variable and  $a_0$ ,  $a_1$  and  $b_0$  are given by

$$\begin{aligned} a_0 &= -2e^{-\alpha\Delta_t} \cos(\beta\Delta_t), \quad a_1 = e^{-2\alpha\Delta_t}, \\ b_0 &= \gamma e^{-\alpha\Delta_t} \sin(\beta\Delta_t) \end{aligned} \quad (17)$$

with

$$\alpha = \frac{\Gamma}{2}, \quad \beta = \sqrt{\omega_0^2 - \alpha^2}, \text{ and } \gamma = \frac{\Delta_t \omega_p^2}{\beta} \quad (18)$$

Using (16) and taking the Z-transform of (7) and (8), the discretization of  $D_z$  and  $B_y$  can be obtained easily by using the Z-transform relation  $\mathcal{Z}^{-m} W(\mathcal{Z}) \rightarrow W^{n-m}$  as

$$D_{z_i}^{n+1} = -a_0 D_{z_i}^n - a_1 D_{z_i}^{n-1} + b_0 E_{z_i}^n \quad (19)$$

$$B_{y_{i+1/2}}^{n+1} = -a_0 B_{y_{i+1/2}}^n - a_1 B_{y_{i+1/2}}^{n-1} + b_0 H_{y_{i+1/2}}^n \quad (20)$$

It should be pointed out that the proposed formulations are applied in the PML regions at the domain boundaries. In non-PML regions, i.e., in the inner FDTD domain, it is only required to set the coefficients  $r_0$  and  $r_1$ , defined in (14), to unity.

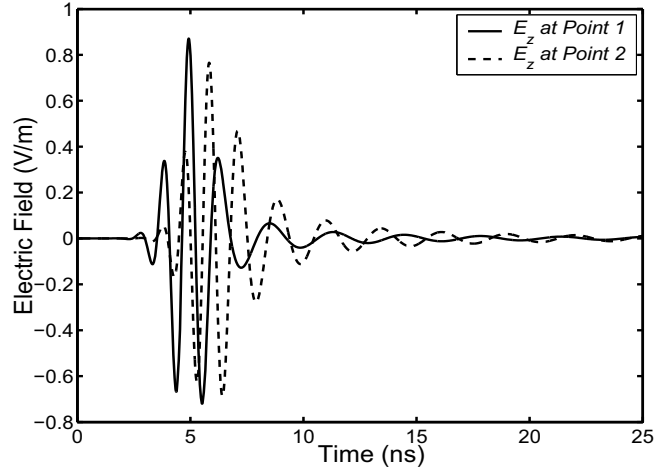


Fig. 1. Early time domain response of  $E_z$  as observed 20 cells (Point 1) and 40 cells (Point 2) away from the excitation point.

### III. SIMULATION STUDY

To show the validity of the proposed formulations, a  $z$ -polarized Gaussian pulse, with a central radial frequency equals to  $\omega_c = 5.0 \times 10^9 \text{ rad/s}$ , was excited at the center of 1-D computational domain which extends in the  $x$  direction and entirely composed of metamaterial realized with a Lorentz model [4] with the parameters of  $\omega_0 = 1.0 \times 10^9 \text{ rad/s}$ ,  $\omega_p = 6.9282 \times 10^9 \text{ rad/s}$  and  $\Gamma = 0$ . With these choices,  $\varepsilon_r(\omega) = \mu_r(\omega) = -1$  at the peak of the incident spectrum  $\omega_c$ . The size of the computational domain was  $100\Delta x$  where  $\Delta x = 5 \text{ mm}$ . In this test, the time step was chosen as  $\Delta_t = 8.33 \text{ ps}$  and the simulation was carried out for the first 32768 time steps.

Figure 1 show the time domain response of the  $E_z$  field recorded at two points: point 1 is located 20 cells and point 2 is located 40 cells away from the excitation point. Both ends of the computational domain were terminated by eight additional PML layers with a quadratic conductivity profile and with 0.001% theoretical reflection coefficient, i.e., PML[8, 2, 0.001%], as defined in [9]. As can be seen from Fig. 1, the early time response of the  $E_z$  field at point 1 occurs before the response at point 2. This confirms that the metamaterial satisfies the causality principle [3]. On the other hand, it is interesting to note that the response at point 2 starts to lead the response at point 1 in the late time as can be seen in Fig. 2 (an expanded view of Fig. 1 in the late time). This is due to the negative index properties of the metamaterial [3].

To measure the reflection performance of the proposed formulations, a reference field is needed. In this paper, the reference field was computed by using a larger domain with the size of  $1000\Delta x$  and truncated by 32 PML layers with the parameters of PML[32, 4, 0.001%]. Figure 3 shows the relative reflection error of the proposed PML formulations as observed one cell away from the PML/computational domain interface for four and eight PML layers with the parameters of PML[4, 2, 0.001%] and PML[8, 2, 0.001%]. The reflection

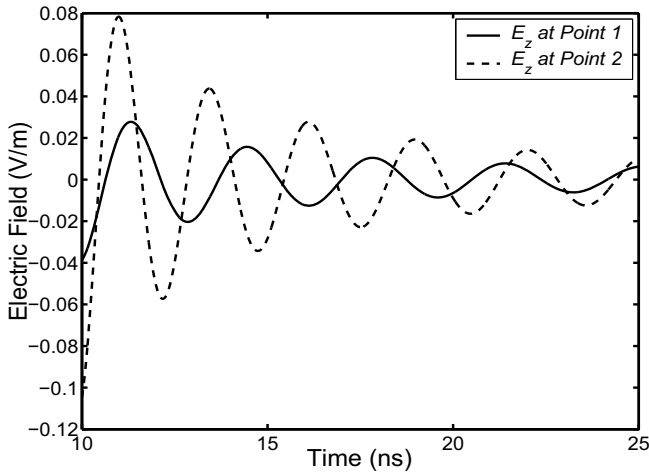


Fig. 2. Late time domain response of  $E_z$  as observed 20 cells (Point 1) and 40 cells (Point 2) away from the excitation point.

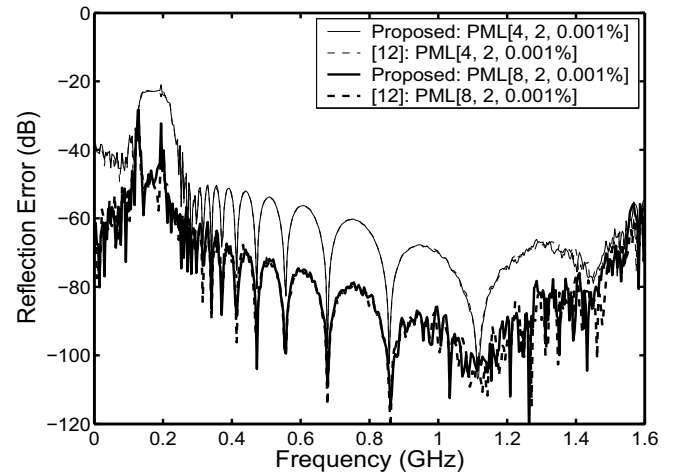


Fig. 4. Reflection coefficient of the proposed PML formulations as a function of frequency.

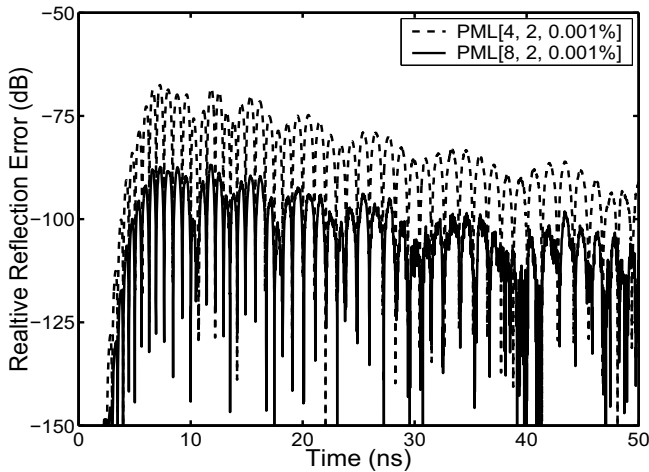


Fig. 3. Relative reflection error of the proposed PML as a function of time.

error was computed as

$$\mathcal{R}_{dB} = 20 \log_{10} \left( \frac{|E_z^R(t) - E_z^T(t)|}{\max[|E_z^R(t)|]} \right) \quad (21)$$

where  $E_z^T(t)$  is the field computed using the test domain, and  $E_z^R(t)$  is the reference field computed using the larger domain. It is apparent from Fig. 3 that good absorbing performance of the proposed formulations has been achieved. Finally, the reflection performance of the proposed formulations as function of frequency was studied. Figure 4 shows the frequency spectrum of the PML reflection coefficient as obtained by using the proposed formulations and by the PML formulations of [12]. The reflections coefficient was observed one space cell away from the PML/computational domain interface and computed as

$$\mathcal{R}_{dB}(f) = 20 \log_{10} \left| \frac{\mathcal{F}\{E_z^R(t) - E_z^T(t)\}}{\mathcal{F}\{E_z^R(t)\}} \right| \quad (22)$$

where  $\mathcal{F}\{\cdot\}$  is the Fourier transform operation. Similar to results reported a single above, the proposed formulations

give good absorbing performance over the frequency range of interest. Furthermore, it should be mentioned that the results of the proposed formulations are very similar to the results obtained by using the PML formulations introduced in [12]. Also, it must be noted that there a sharp increase in the reflection coefficient in the frequency range of 0.1-0.3 GHz, but this is mainly due to the resonance frequency of the metamaterial, i.e.,  $f_0 = \omega_0/2\pi = 0.15915GHz$ . Finally, it should be mentioned that the reflections start to increase for frequencies beyond 1.60 GHz. This is because the pulse used in this paper does not contain significant frequency components at those frequencies.

#### IV. CONCLUSION

In this paper, efficient and simple PML formulations are presented for truncating metamaterial computational domains. In the proposed formulations, the frequency dependence of the metamaterials has been implemented in the FDTD algorithm by using the Z-transform theory. Numerical example carried out in one dimensional domain composed entirely of Lorentzian type metamaterial show that good absorbing performance has been obtained with the proposed PML formulations. The formulations can be extended to 2-D and 3-D cases in a similar manner.

#### REFERENCES

- [1] V. G. Veselago, "The electrodynamics of substances with simultaneously negative values of  $\epsilon$  and  $\mu$ ," *Sov. Phys. Uspekhi*, vol. 10, no. 4, pp. 509-514, 1968.
- [2] A. Taflov and S.C. Hangess, *Computational electrodynamics: The Finite-Difference Time-Domain Method*, 2nd ed. Boston, MA: Artech House, 2000.
- [3] R. W. Ziolkowski and E. Heyman, "Wave propagation in media having negative permittivity and permeability," *Phys. Rev. E*, vol. 64, 056625, 2001.
- [4] M. K. Kärkkäinen, "Numerical study of wave propagation in uniaxially anisotropic Lorentzian backward-wave slabs," *Phys. Rev. E*, vol. 68, 026602, 2003.
- [5] M. K. Kärkkäinen and S. I. Maslovski, "Wave propagation, refraction, and focusing phenomena in lorentzian double-negative materials: A theoretical and numerical study," *Microwave and Optical Technology Letters*, vol. 37, No. 1, pp. 4-7, 2003.

- [6] M. M Suwailam, and Z. Chen, "Simulation of lorentzian DNG materials using FDTD methods and Z-transform," 20th Annual Review of Pogress in Applied Computational Electromagnetics, Syracuse, NY, April 19-23, 2004.
- [7] A. Akyurtlu, and D. H. Werner, "A novel dispersive FDTD formulation for modeling transient propagation in chiral metamaterials," IEEE Transactions on Antennas and Propagation, vol. 52, no. 9, pp. 2267-2276, 2004.
- [8] M. W. Feise, J. B. Schneider, and P. J. Bevelacqua, "Finite-difference and pseudospectral time-domain methods applied to backwards-wave metamaterials," IEEE Transactions on Antennas and Propagation, vol. 52, no. 11, pp. 2955 - 2962, 2004.
- [9] J.-P. Berenger, "A perfectly matched layer for the absorption of electromagnetic waves," J. Comput. Phys., vol. 114, pp. 185-200, 1994.
- [10] D. Correia and J.-M. Jin, "3D-FDTD-PML analysis of left-handed metamaterials," Microwave Optical Technol. Lett., vol. 40, no. 3, pp. 201-205, 2004.
- [11] S. A. Cummer, "Perfectly matched layer behavior in negative refractive index materials," IEEE Antennas Wireless Propagat. Lett., vol. 3, pp. 172-175, 2004.
- [12] X. T. Dong, X. S. Rao, Y. B. Gan, B. Guo and W. Y. Yin, "Perfectly matched layer-absorbing boundary condition for left-handed materials," IEEE Microwave Wireless Compon. Lett., vol. 14, no. 6, pp. 301-303, 2004.
- [13] D. M. Sullivan, "Frequency-dependent FDTD methods using Z transforms," IEEE Trans. Antennas Propag., vol. 40., no. 10, pp. 1223-1230, 1992.